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SHEAR DEFORMATION PLATE CONTINUA OF LARGE DOUBLE LAYERED SPACE STRUCTURES†

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Abstract—A simple method is presented to model large rigid-jointed lattice structures as continuous elastic media with couple stresses using energy equivalence. In our analysis the transition from the discrete system to the continuous media is achieved by expanding the displacements and the rotations of the nodal points in a Taylor series about a suitable chosen origin. The strain energy of the continuous media with couple stresses is then specialized to obtain shear deformation plate continua. Equivalent continua for single layered grids, double layered grids and three-dimensional lattices are then obtained.

1. INTRODUCTION

The last decade has witnessed a dramatic increase in research activities dealing with the possibility of utilizing space for various commercial and scientific needs. Large lattice-type structures are analyzed as candidates to meet such applications. In order to assess the utility of such structures, complete understanding of their mechanical and thermal behavior is needed. Continuum approximations provide practical means for achieving this understanding.

In three recent papers[1–3] we derived the stiffness coefficients of equivalent continua by using a building block approach consisting of obtaining the effective properties of the smallest unit cell of the repeating structure and then using orthogonal transformation techniques to obtain the overall properties. In Refs [1, 2] we constructed the equivalent continuum for discrete pin-jointed repetitive structures using the rod's unidirectional property as our building block unit. In a more recent paper[3] we derived the effective properties of rigid jointed (frame type) repetitive structures. This differed substantially from the truss-like structures in that the influence of in-plane bending rigidities to the structure are included. The fact that the individual rod in a rigid-jointed array can resist in-plane bending dictated that the smallest sub-cell of the structure which was used as the building block was no longer unidirectional and thus had to be two-dimensional substructure. The most degenerate basic two-dimensional frame structures were found to be the (0°, 90°) and the (0°, ±60°) layups. Effective properties for these sub-cells were constructed using simple strength of materials and approaches such as the matrix structural analysis methods[4–6]. This resulted in two-dimensional generalization of the one-dimensional area weighted properties needed in the analysis of pin-jointed structures[1, 2]. The analysis of Refs [1–3] gave exact results for the constitutive relations of one-dimensional (unidirectional) layups; "strictly" two-dimensional (in the forms of (0°, 90°), (0°, ±60°)) layups and the "strictly" three-dimensional layups as we defined in Ref. [2].

By invoking the classical plate theory assumptions the results of the strictly three-

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dimensional structures were successfully reduced to those of the quasi-three-dimensional problem[2] only for the case of pin-jointed structures. Here a quasi-three-dimensional model consists of two surface sheets separated by a thickness h and are connected by diagonals to form a plate. In attempting to obtain the equivalent plate continuum for a discrete frame-like structure, expressions for its general bending rigidities could not be obtained by the techniques described in Refs [1–3]. Therefore, we need to use another approach to develop these requirements. The energy equivalence technique presents a way of developing such properties. As a by-product of using the energy equivalence approach, we obtain, besides the bending rigidities, the information for the stiffness coefficients. The results obtained by the energy method will also help in assessing the accuracy of the building block approach described in Ref. [3].

In this paper we use the energy equivalence to construct equivalent continua for the actual lattice structure. An energy equivalent continuum is defined as that which has the same amount of strain and kinetic energies stored in it as that of the original lattice structure when both are subjected to the same loading conditions. The equivalent continuum is characterized by its strain and kinetic energies from which the constitutive relations and the equations of motion can be derived.

The basic concept in the energy approach is the existence of kinematic variables which are functions of continuous spatial coordinates as opposed to those in the lattice theory which are defined at discrete points of members. To relate both the discrete and continuum models, a Taylor series expansion has been commonly used in constructing the equivalent continuum.

Previous studies which utilize the energy equivalence approach, such as those carried out by Sun and Yang[7], Noor *et al.*[8, 9], Bazant and Christensen[10, 11], and Nemeth[12] are available in the literature. Bazant and Christensen derived an equivalent micro-polar continuum for large grid frameworks in order to solve the extensional buckling of a multi-story, multi-bay rectangular frame. Sun and Yang[7] established a two-dimensional in-plane continuum model with couple stress for a (0°, 90°) layup. Noor *et al.* [8] constructed the equivalent continuum of a double layer grid assuming all joints to be pinned. Also, Noor and Nemeth[13] developed micropolar models for large repetitive beam-like planar lattices with rigid joints. Nemeth[12] derived the strain energy of the single layer grids with rigid joints in terms of strains and curvatures of its beam members. These are then expressed in terms of the strains and curvatures of the continuum.

The present study utilizes the energy approach to present a simple method to model large rigid-jointed lattices as continuous media with couple stresses. In our analysis the transition from the discrete system to the continuous medium is achieved by expanding the displacements and the rotations of the nodal points in a Taylor series about a suitable chosen origin. Here basic kinematic assumptions are introduced to insure that the assumptions used in deriving the governing equations of the modeled continuum are satisfied. Accordingly, the number of terms retained in the Taylor series expansion will depend upon the properties to be evaluated. This implies that one has first to determine what kind of continuum is needed to model from the discrete lattice, before the actual properties are derived.

In Section 2 we present our analysis followed in Section 3 by a comparison between our approach and those reported in the literature. Finally, in Section 4 we present a variety of applications.

2. ANALYSIS

2.1. *Linearized constitutive equations for elastic materials with couple stresses*

The internal energy of an elastic material without couple stresses may be expressed as a function of the material strain tensor. Toupin[14] has shown that when couple stresses are taken into consideration, the energy function will be a homogeneous quadratic function of the material strain tensor ε_{ij} and the curvature twist tensor κ_{ij} which are defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

$$\kappa_{ij} = \frac{1}{2} e_{jlk} u_{k,li} \tag{2}$$

where u_i are the components of the displacement vector and e_{jlk} is the permutation symbol.

The strain energy function describing the general constitutive equations for a linear elastic material with couple stresses can be obtained as[15]

$$W = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + B_{ijkl}\varepsilon_{ij}\kappa_{kl} + \frac{1}{2} E_{ijkl}\kappa_{ij}\kappa_{kl} \tag{3}$$

where W is the strain energy function. This strain energy expression would describe an elastic material without couple stresses when all the B_{ijkl} and E_{ijkl} coefficients vanish.

2.2. *Determination of the characteristics of the equivalent continuum models*

The steps used in the construction of the equivalent continuum are given below.

(1) Isolate the smallest repeating element from the lattice.

(2) Write the stiffness matrix of this repeating element and calculate its strain energy in terms of its nodal displacements and rotations.

(3) The displacements and the rotations of the nodal points are then expanded in a Taylor series about a suitable chosen origin. Basic kinematic assumptions are then introduced to insure that the assumptions used in deriving the governing equations of the modeled continuum are satisfied.

(4) The displacement expansions obtained in step 3 are substituted in the energy expression of the repeating element to obtain the energy expression of the equivalent continuum, from which we can determine the characteristics of the equivalent continuum model.

The strain energy of the repeating element of a lattice with rigid joints is given by[18]

$$U = \sum_{\text{members}} \frac{1}{2}\{\Delta\}^{\mathrm{T}}[\Gamma^{(m)}]^{\mathrm{T}}[K^{(m)}][\Gamma^{(m)}]\{\Delta\} \tag{4}$$

where $\{\Delta\}$ is the vector of nodal displacements and rotations of a typical member, $[K^{(m)}]$ is the elemental stiffness matrix of the typical beam in local coordinates, $[\Gamma^{(m)}]$ is the member transformation matrix in local coordinates, the superscripts m and T denote the mth member in the repeating element and transposition, respectively. As mentioned earlier, the transition from the discrete lattice to the equivalent continuous medium is done by expanding the nodal displacements and rotations about the origin of the repeating element by a Taylor series. The number of terms retained in the Taylor series expansion and the kinematic assumptions used on the continuous displacement and rotation variables will depend upon the properties to be evaluated.

This implies that we have first to determine what kind of continuum we need to model from the discrete lattice: a linear elastic material without couple stresses where the motion is treated as a three-dimensional problem of stress analysis or an equivalent plate continuum where the fundamental equations of the plate are used.

2.3. *Kinematic assumptions used to model a linear elastic material without couple stresses*

Here we evaluate the stiffness coefficients, C_{ijkl}, for the equivalent elastic linear continuum whose strain energy is

$$W = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \qquad i,j,k,l = 1,2,3. \tag{5}$$

This will be constructed from a single layer grid or multi-layer grids. For this case, the nodal displacements of the repeating element are expanded up to the second-order Taylor series expansion, whereas the nodal rotations follow a one-term expansion. This implies that the rotations are considered to be constant for all the nodal points of the repeating element. Hence, for a typical node (x_i, y_i, z_i) these expansions are affected as

$$u_i = u + x_i\frac{\partial u}{\partial x} + y_i\frac{\partial u}{\partial y} + z_i\frac{\partial u}{\partial z} \tag{6a}$$

$$v_i = v + x_i\frac{\partial v}{\partial x} + y_i\frac{\partial v}{\partial y} + z_i\frac{\partial v}{\partial z} \tag{6b}$$

$$w_i = w + x_i\frac{\partial w}{\partial x} + y_i\frac{\partial w}{\partial y} + z_i\frac{\partial w}{\partial z} \tag{6c}$$

$$\theta_{x_i} = \theta_x, \qquad \theta_{y_i} = \theta_y, \qquad \theta_{z_i} = \theta_z \tag{7}$$

where u, v, w, θ_x, θ_y and θ_z are the displacements and rotations of continuous functions which assume the values of the displacements and rotations at the origin of the repeating element. Furthermore, the rotation functions θ_x, θ_y and θ_z are the component rotations defining the rotation of the rigid equivalent continuum, therefore, they are expressed in terms of the displacement functions u, v, w as

$$\theta_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \theta_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \theta_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{8}$$

The strain energy of the equivalent continuum is obtained by substituting the expressions for the displacement and rotation given by eqns (6)–(8) into the expression for the strain energy of the repeating element given in eqn (4). By differentiating according to eqn (5), we obtain the three-dimensional stiffness coefficients of the continuum.

2.4. *Kinematic assumptions used to model shear deformation plate continuum*

The strain energy function for a repeating element governed by the shear deformation plate theory is given in the appendix, eqn (A1)[9]. Notice that the plate curvature, $K_{\alpha\beta}$, are components of the general curvature-twist tensor, κ_{ij}, defined in eqn (2); specifically, we have

$$\kappa_{12} \to K_{11}, \qquad \kappa_{21} \to K_{22}, \qquad 2(\kappa_{22} - \kappa_{11}) \to K_{12}. \tag{9}$$

Therefore, and by examining the governing equations of the shear deformation plate continuum, we establish the following procedure, using the superposition principle, to evaluate the different characteristic coefficients for that continuum.

(1) Evaluate all the A_{ijkl} stiffness coefficients as if the equivalent continuum was a three-dimensional linear elastic medium with couple stresses; the stiffness coefficients $A_{\alpha\beta\gamma\rho}$ of the reduced model are determined in terms of A_{ijkl} as per eqn (A4) in the appendix. The stiffness coefficients A_{ijkl} are defined as

$$A_{ijkl} = hC_{ijkl} \tag{10}$$

where h is the plate thickness. (Notice that: $i, j, k, l = 1, 2, 3$, and $\alpha, \beta, \gamma, \rho = 1, 2$.)

(2) Evaluate the bending stiffness coefficients D_{ijkl} and the coupling coefficients $F_{\alpha\beta\gamma\rho}$ of the equivalent continuum as if it was governed by the constitutive equations classical plate theory. The stiffness coefficients $A_{\alpha 3\beta 3}$ and $2A_{\alpha 3\beta\gamma}$ which represent the shear deformation contribution to the governing equations of the plate continuum are determined

from the first step. The coupling and bending coefficients of the reduced model are then evaluated using eqns (A5) and (A6) in the appendix.

Specifically, we determine the strain energy expression for the equivalent continuum as if it is a linear elastic material with couple stresses, as required by Toupin's constitutive equations. This is followed by specializing this strain energy expression to obtain the corresponding one for an equivalent plate continuum using the same assumptions used to obtain the governing equations for the plate continuum from the governing equations for the linear elastic continuum with couple stresses. This is done by retaining the following expansion forms

$$u_i = u + x_i\frac{\partial u}{\partial x} + y_i\frac{\partial u}{\partial y} + z_i\frac{\partial u}{\partial z} + \frac{1}{2}\left(y_i^2\frac{\partial^2 u}{\partial y^2} + z_i^2\frac{\partial^2 u}{\partial z^2} + 2x_iy_i\frac{\partial^2 u}{\partial x\partial y} + 2y_iz_i\frac{\partial^2 u}{\partial y\partial z} + 2x_iz_i\frac{\partial^2 u}{\partial x\partial z}\right) \tag{11a}$$

$$v_i = v + x_i\frac{\partial v}{\partial x} + y_i\frac{\partial v}{\partial y} + z_i\frac{\partial v}{\partial z} + \frac{1}{2}\left(x_i^2\frac{\partial^2 v}{\partial x^2} + z_i^2\frac{\partial^2 v}{\partial z^2} + 2x_iy_i\frac{\partial^2 v}{\partial x\partial y} + 2y_iz_i\frac{\partial^2 v}{\partial y\partial z} + 2x_iz_i\frac{\partial^2 v}{\partial x\partial z}\right) \tag{11b}$$

$$w_i = w + x_i\frac{\partial w}{\partial x} + y_i\frac{\partial w}{\partial y} + z_i\frac{\partial w}{\partial z} + \frac{1}{2}\left(x_i^2\frac{\partial^2 w}{\partial x^2} + y_i^2\frac{\partial^2 w}{\partial y^2} + 2x_iy_i\frac{\partial^2 w}{\partial x\partial y} + 2y_iz_i\frac{\partial^2 w}{\partial y\partial z} + 2x_iz_i\frac{\partial^2 w}{\partial x\partial z}\right) \tag{11c}$$

$$\theta_{x_i} = \theta_x + x_i\frac{\partial \theta_x}{\partial x} + y_i\frac{\partial \theta_x}{\partial y} + z_i\frac{\partial \theta_x}{\partial z} \tag{12a}$$

$$\theta_{y_i} = \theta_y + x_i\frac{\partial \theta_y}{\partial x} + y_i\frac{\partial \theta_y}{\partial y} + z_i\frac{\partial \theta_y}{\partial z} \tag{12b}$$

$$\theta_{z_i} = \theta_z + x_i\frac{\partial \theta_z}{\partial x} + y_i\frac{\partial \theta_z}{\partial y} + z_i\frac{\partial \theta_z}{\partial_z} \tag{12c}$$

where u, v, w, θ_x, θ_y and θ_z are the displacements and rotations of continuous functions which assume the values of the displacements and the rotations at the origin of the repeating element. The rotation functions θ_x, θ_y and θ_z are expressed in terms of the displacement variables u, v and w as per eqn (8).

The strain energy of the equivalent linear elastic continuum with couple stresses is obtained by substituting the above expressions for the displacements and rotation expansions into the expression of the strain energy of the repeating element given in eqn (4). In order to obtain the equivalent classical continuum plate from the linear elastic media with couple stresses, one has to impose the following two assumptions on the expression of the strain energy of that media: firstly, by assuming bending to occur in the x–y plane only, some terms in the curvature-twist tensor of the elastic media do not contribute to the strain energy of that model and will be nonexistent. These terms are

$$\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x\partial y}, \frac{\partial^2 w}{\partial y\partial z}, \frac{\partial^2 w}{\partial x\partial z}.$$

Secondly, following the classical plate theory assumption, the transverse shear strains ε_{23} and ε_{13} are negligible. These lead to the following constraints

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}. \tag{13}$$

The expressions for the rotation functions thus become

$$\theta_x = \frac{\partial w}{\partial y}, \qquad \theta_y = -\frac{\partial w}{\partial x}. \tag{14}$$

The plate curvatures are expressed as[16]

$$K_{11} = -\frac{\partial^2 w}{\partial x^2}, \qquad K_{22} = -\frac{\partial^2 w}{\partial y^2}, \qquad K_{12} = -\frac{\partial^2 w}{\partial x \partial y}. \tag{15}$$

Using relations (13) and (14) into the expression of the strain energy of the linear elastic media with couple stresses we end with the strain energy of the classical plate continuum.

3. COMPARISON WITH OTHER ENERGY METHODS

At this point, we would like to compare our energy equivalence approach for continuum modeling of the large discrete structures with those reported in the literature.

(1) Nemeth[12] derived the strain energy of the single layer grid in terms of its beam member's strains and curvatures, and consequently expressed it in terms of the strains and curvatures of the continuum.

(2) Noor *et al.*[9] derived the equivalent continuum for double layer grids with pinned joints. The strain energy of the plate continuum was obtained by replacing the axial strain in each member of the repeating element by its expression in terms of the strain components in the coordinate directions evaluated at the center of each member; and then expanding these strain components in Taylor series about a suitably chosen origin.

(3) Noor *et al.*[8] derived the equivalent continuum for double layer grids with pinned joints using an approach similar to the one proposed here. In their theory, a linear variation in the normal coordinate z was assumed for the displacement components; the nodal displacements were then expanded in a two-term Taylor series expansion. Therefore, at a typical node (x_i, y_i, z_i) of the repeating element they obtained, as an example, the following expression for the nodal displacement, w_i in the z direction

$$w_i = \left(w + z_i\frac{\partial w}{\partial z}\right) + x_i\left(\frac{\partial w}{\partial x} + z_i\frac{\partial^2 w}{\partial x \partial z}\right) + y_i\left(\frac{\partial w}{\partial y} + z_i\frac{\partial^2 w}{\partial y \partial z}\right) \tag{16a}$$

and rearranging terms they obtained

$$w_i = w + \left(x_i\frac{\partial w}{\partial z} + y_i\frac{\partial w}{\partial y} + z_i\frac{\partial w}{\partial z}\right) + \left(x_i z_i\frac{\partial^2 w}{\partial x \partial z} + y_i z_i\frac{\partial^2 w}{\partial y \partial z}\right). \tag{16b}$$

The difference between both approaches can be seen by comparing eqn (16b) with eqn (11c).

Here we would like to add that it is easier to write the strain energy of the repeating element in terms of its nodal displacements than writing it in terms of its beam member's strains and curvatures; furthermore, the modeling of linear elastic media with couple stresses for large lattice structures with rigid joints has not been presented before.

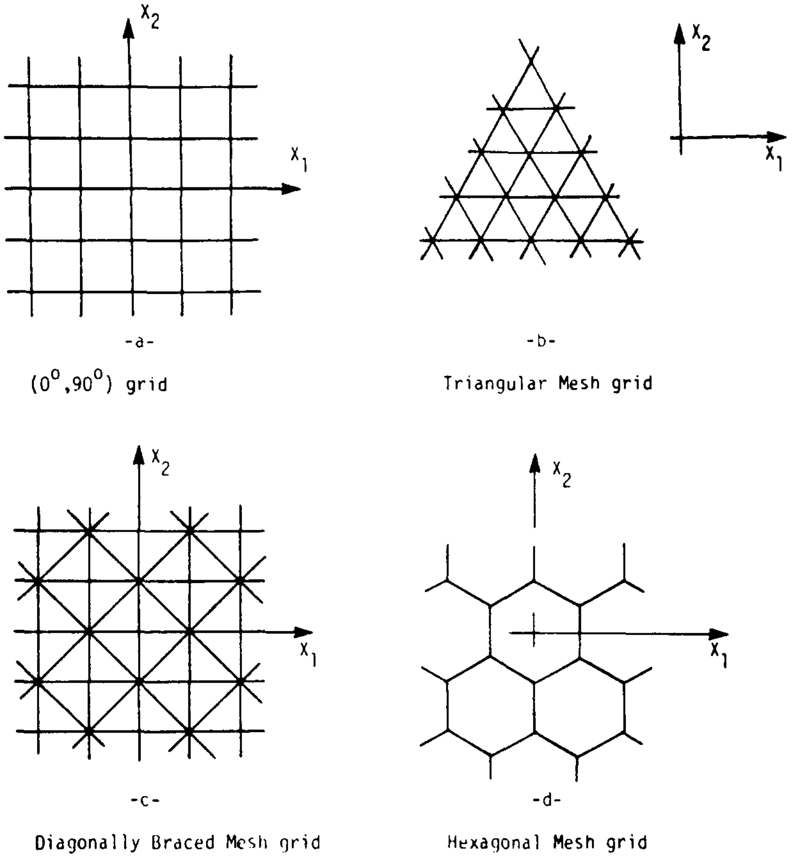

Fig. 1. Four different single layer lattice grids.

4. APPLICATIONS

4.1. *Single layer grids*

In this section an application of the energy method to determine the characteristics of the equivalent plate continuum of single layer grids is presented. The grids are considered to be rigidly connected and to have both bending and torsional rigidities. We notice here that the terms containing $(\partial/\partial z)$ and $(\partial^2/\partial z^2)$ in the expansion of the nodal displacements and rotations do not appear since all the grid joints lie in the same plane.

The repeating element for the (0°, 90°) grid, the triangular mesh grid, the diagonally braced mesh grid and the hexagonal mesh grid (Figs 1(a)–(d)) at any arbitrary point are shown in Figs 2(a)–(d). The areas of the repeating element for these mesh grids are L^2, $(\sqrt{3}L^2)/2$, $2L^2$ and $(3\sqrt{3}L^2)/2$, respectively.

The stiffness coefficients and the bending rigidities of the equivalent plate continuum for the (0°, 90°), (0°, ±60°) and (0°, 90°, ±45°) lattices are given in Table 1. The stiffness coefficients and the bending rigidities characterizing the equivalent plate continuum for the hexagonal planar lattice are found to be one third of those corresponding to the (0°, ±60°) layup. This result, obtained using the energy equivalence, confirms the results obtained in Ref. [17] using the "building block" approach.

4.2. *Three-dimensional structures and double layered structures*

In our analysis we shall differentiate between the three-dimensional structures and the double layered structures. The representative candidate in our study is the octetruss structure[18]. We shall first obtain the properties of its equivalent linear elastic continuum without couple stresses. After that, we shall model the plate continuum of the double layered tetrahedral grid (which is the quasi-three-dimensional model of the octetruss structure).

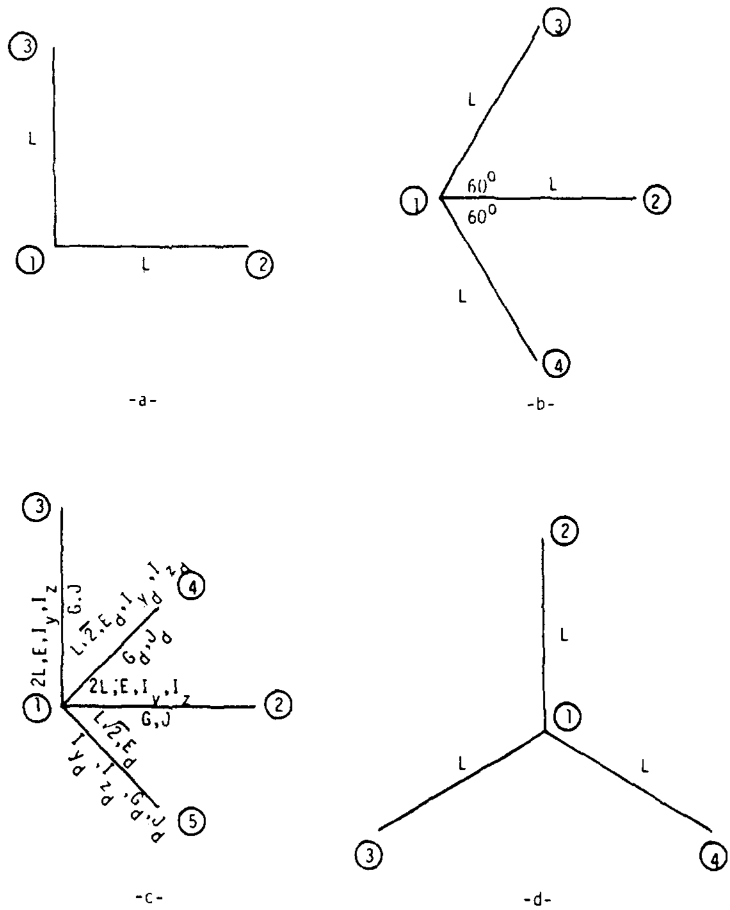

Fig. 2. Repeating elements for the (0°, 90°), (0°, ±60°), (0°, 90°, ±45°) and the hexagonal grids.

Table 1. Stiffness coefficients for plate continuum from single layered grids

	(0°, 90°) Grid	(0°, ±60°) Grid	(0°, 90°, ±45°) Grid
$A_{1111} = A_{2222}$	$\frac{EA}{L}$	$\frac{3\sqrt{3}}{4}\frac{EA}{L} + 3\sqrt{3}\frac{EI_z}{L^3}$	$\frac{EA}{L} + \frac{\sqrt{2}}{4}\frac{E_d A_d}{L} + \frac{3\sqrt{2}}{2}\frac{E_d I_{z_d}}{L^3}$
A_{1122}	0	$\frac{\sqrt{3}}{4}\frac{EA}{L} - 3\sqrt{3}\frac{EI_z}{L^3}$	$\frac{\sqrt{2}}{4}\frac{E_d A_d}{L} - \frac{3\sqrt{2}}{2}\frac{E_d I_{z_D}}{L^3}$
A_{1212}	$\frac{6EI_z}{L^3}$	$\frac{\sqrt{3}}{4}\frac{EA}{L} + 3\sqrt{3}\frac{EI_z}{L^3}$	$\frac{6EI_z}{L^3} + \frac{\sqrt{2}}{4}\frac{E_d A_d}{L}$
$A_{1313} = A_{2323}$	$\frac{3EI_y}{L^3}$	$3\sqrt{3}\frac{EI_y}{L^3}$	$\frac{3EI_y}{L^3} + \frac{3\sqrt{2}}{4}\frac{E_d I_{y_d}}{L^3}$
$D_{1111} = D_{2222}$	$\frac{EI_y}{L}$	$\frac{3\sqrt{3}}{4}\frac{EI_y}{L} + \frac{\sqrt{3}}{4}\frac{GJ}{L}$	$\frac{EI_y}{L} + \frac{\sqrt{2}}{4}\frac{E_d I_{y_d}}{L} + \frac{\sqrt{2}}{4}\frac{G_d J_d}{L}$
D_{1122}	0	$\frac{\sqrt{3}}{4}\frac{EI_y}{L} - \frac{\sqrt{3}}{4}\frac{GJ}{L}$	$\frac{\sqrt{2}}{4}\frac{E_d I_{y_d}}{L} - \frac{\sqrt{2}}{4}\frac{G_d J_d}{L}$
D_{1212}	$\frac{GJ}{2L}$	$\frac{\sqrt{3}}{4}\frac{EI_y}{L} + \frac{\sqrt{3}}{4}\frac{GJ}{L}$	$\frac{GJ}{2L} + \frac{\sqrt{2}}{4}\frac{E_d I_{y_d}}{L}$

4.2.1. *Three-dimensional octetruss structure.* This structure is shown in Fig. 3. We shall assume that all the beams have the same geometric properties. In view of the periodic nature of the structuring, we shall focus attention on joint (x_i, y_i, z_i). A typical beam assembly element at this point is displayed in Fig. 4.

In order to derive the effective stiffness properties of this repeating element, we have

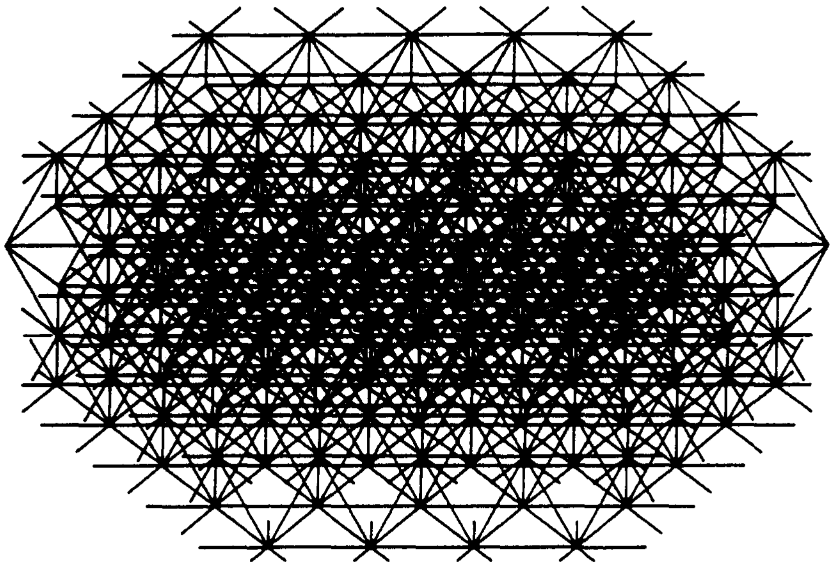

Fig. 3. Three-dimensional octetruss structure.

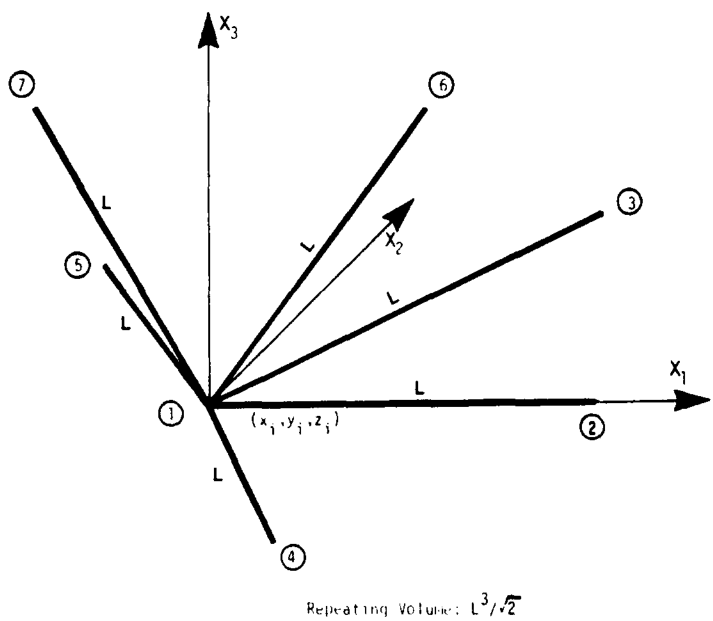

Fig. 4. Repeating element of the octetruss structure.

to determine the directions of the principal axes of the cross sections of its beam elements, 0y and 0z, with respect to the fixed directions X_1, X_2 and X_3. The member transformation matrix in local coordinates, $[\Gamma]$, (of order 12 × 12) is given as[16]

$$[\Gamma] = \begin{bmatrix} [T] & & & \\ & [T] & & \\ & & [T] & \\ & & & [T] \end{bmatrix} \tag{17}$$

where $[T]$ represents the matrix of direction cosines of the 0x, 0y and 0z directions, respectively. It is measured in the global system X_1, X_2, and X_3, and is given by

$$[T] = \begin{bmatrix} l_{0x} & m_{0x} & n_{0x} \\ l_{0y} & m_{0y} & n_{0y} \\ l_{0z} & m_{0z} & n_{0z} \end{bmatrix}. \tag{18}$$

For the repeating element of Fig. 3, the octetruss is considered to be constructed from three single squared layers having a different orientation in space. The local 0z principal axis of each beam is defined to be the one which is perpendicular to the single layer grid containing that particular beam. Therefore, and with reference to Fig. 3, the coordinates of the seven nodes of the repeating element are given in Table 2.

Table 2

Joint	X_1	X_2	X_3
1	0	0	0
2	L	0	0
3	$L/2$	$L\sqrt{3}/2$	0
4	$L/2$	$-L\sqrt{3}/2$	0
5	0	$-L\sqrt{3}/3$	$L\sqrt{2}/\sqrt{3}$
6	$L/2$	$L\sqrt{3}/6$	$L\sqrt{2}/\sqrt{3}$
7	$-L/2$	$L\sqrt{3}/6$	$L\sqrt{2}/\sqrt{3}$

The matrices of the direction cosines, $[T]$, for the different beam members in this element are given by

$$[T]_{1\text{-}2} = \begin{matrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ 0 & -\sqrt{2}/\sqrt{3} & -1/\sqrt{3} \end{matrix}$$

$$[T]_{1\text{-}3} = \begin{matrix} 1/2 & \sqrt{3}/2 & 0 \\ -1/2 & \sqrt{3}/6 & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{matrix}$$

$$[T]_{1\text{-}4} = \begin{matrix} 1/2 & -\sqrt{3}/2 & 0 \\ -1/2 & -\sqrt{3}/6 & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{matrix}$$

$$[T]_{1\text{-}5} = \begin{matrix} 0 & -1/\sqrt{3} & 2/\sqrt{3} \\ -1 & 0 & 0 \\ 0 & -\sqrt{2}/\sqrt{3} & -1/\sqrt{3} \end{matrix}$$

$$[T]_{1\text{-}6} = \begin{matrix} 1/2 & \sqrt{3}/6 & \sqrt{2}/\sqrt{3} \\ 1/2 & -\sqrt{3}/2 & 0 \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{matrix}$$

$$[T]_{1\text{-}7} = \begin{matrix} -1/2 & \sqrt{3}/6 & \sqrt{2}/\sqrt{3} \\ -1/2 & -\sqrt{3}/2 & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{matrix}$$

The analysis described in Section 2 is carried out. The strain energy of the repeating element is evaluated; the nodal displacements and rotations are expanded according to

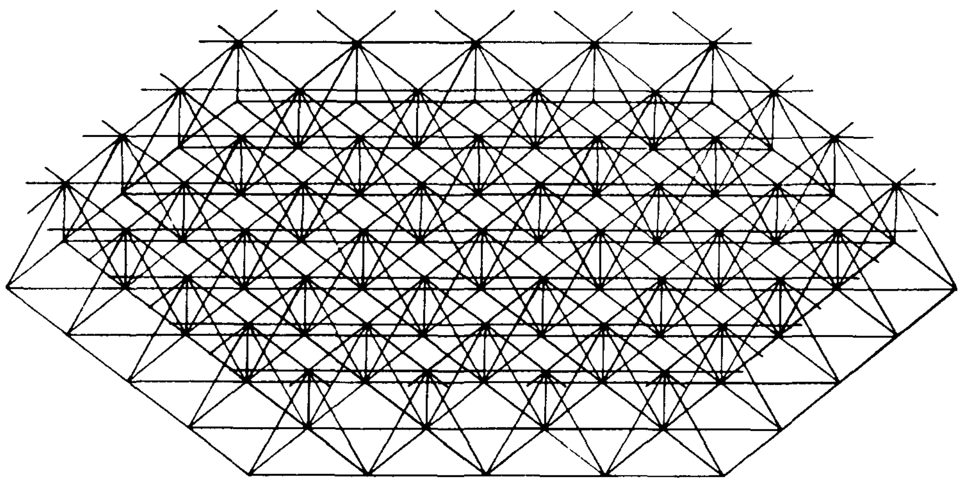

Fig. 5. Quasi-three-dimensional octetruss structure corresponding to Fig. 3.

the two- and one-term expansion, respectively, with respect to the nodal joint (x_i, y_i, z_i) as the suitable chosen origin; the continuous rotation functions are expressed according to eqn (8) and finally the effective properties of the equivalent elastic linear continuum without couple stresses are evaluated. These are found identical to those reported in Refs [3, 17].

We have confirmed[17] that for the repeating element of Fig. 4, the structure can be considered to be constructed from four (0°, ±60°) layups with the restriction of using circular cross-sectional beams. This was actually done and the results were found to be identical to those reported in Ref. [17].

4.2.2. *Double layered grids.* The double layered grids are also known to be the quasi-three-dimensional structures[2]). Here, the double layered tetrahedral grid is studied. It consists of two parallel layers of (0°, ±60°) beams connected by diagonal members which form three-sided pyramids as shown in Fig. 5. In this double layered grid, all the members have the same length L. In order to differentiate between the role of the upper and lower chords and the diagonals, we shall assume that the two layers and the diagonals have different geometric and material properties from each other. If the upper and lower layers and the diagonals are designated by the subscripts 1, 2 and d, respectively, their geometric properties will be designated by $(A_1, I_{y1}, I_{z1}, J_1)$, $(A_2, I_{y2}, I_{z2}, J_2)$ and $(A_d, I_{yd}, I_{zd}, J_d)$, respectively, while their material properties are designated by (E_1, G_1), (E_2, G_2) and (E_d, G_d), respectively.

Since we intend to derive the characteristics of the equivalent plate continuum for this quasi-three-dimensional structure, we must have the origin of its repeating element at the middle of the distance separating its upper and lower layers. The equivalent plate continuum for this structure is derived as if it was constructed from three different (0°, 90°) single layers. The area of the repeating element of the double layered grid shown in Fig. 6 is $(\sqrt{3}L^2)/2$.

The strain energy of the repeating element is evaluated. The nodal displacements and rotations are expanded according to the three- and two-term expansions, respectively; the continuous rotation functions are expressed according to eqn (8); the assumptions of the classical plate theory are introduced and finally, the effective properties of the equivalent plate continuum are evaluated. Those properties are listed in Table 3.

5. SUMMARY AND CONCLUSIONS

Energy and equivalence techniques were used to construct equivalent continuous elastic media with couple stresses for large rigid-jointed lattice structures. The strain energy of these media was then specialized to obtain the equivalent shear deformation plate

Table 3. Stiffness coefficients for plate continuum of the double layered tetrahedral grids constructed using three single layers

$A_{1111} = A_{2222}$	$\dfrac{9\left(E_1A_1 + E_2A_2 + \frac{1}{9}E_dA_d\right)}{4\sqrt{3}L} + \dfrac{6(E_1I_{y_1} + E_2I_{y_2} + E_dI_{y_d})}{\sqrt{3}L^3} + \dfrac{\sqrt{3}(E_1I_{z_1} + E_2I_{z_2} + E_dI_{z_d})}{L^3}$
A_{1122}	$\dfrac{3\left(E_1A_1 + E_2A_2 + \frac{1}{9}E_dA_d\right)}{4\sqrt{3}L} - \dfrac{6\left(E_1I_{y_1} + E_2I_{y_2} - \frac{1}{3}E_dI_{y_d}\right)}{\sqrt{3}L^3} - \dfrac{\sqrt{3}(E_1I_{z_1} + E_2I_{z_2} + E_dI_{z_d})}{L^3}$
A_{1212}	$\dfrac{3\left(E_1A_1 + E_2A_2 + \frac{1}{9}E_dA_d\right)}{4\sqrt{3}L} + \dfrac{6\left(E_1I_{y_1} + E_2I_{y_2} + \frac{1}{3}E_dI_{y_d}\right)}{\sqrt{3}L^3} + \dfrac{\sqrt{3}(E_1I_{z_1} + E_2I_{z_2} + E_dI_{z_d})}{L^3}$
$A_{1133} = A_{2233}$	$\dfrac{2E_dA_d}{3\sqrt{3}L} - \dfrac{8E_dI_{y_d}}{\sqrt{3}L^3}$
A_{3333}	$\dfrac{8E_dA_d}{3\sqrt{3}L} + \dfrac{16E_dI_{y_d}}{\sqrt{3}L^3}$
$A_{1123} = -A_{2223}$ $= A_{1312}$	$\dfrac{\sqrt{2}E_dA_d}{6\sqrt{3}L} + \dfrac{\sqrt{6}\left(E_1I_{y_1} + E_2I_{y_2} + \frac{1}{3}E_dI_{y_d}\right)}{L^3} - \dfrac{\sqrt{6}(E_1I_{z_1} + E_2I_{z_2} + E_dI_{z_d})}{L^3}$
$A_{2323} = A_{1313}$	$\dfrac{2E_dA_d}{3\sqrt{3}L} + \dfrac{\sqrt{3}\left(E_1I_{y_1} + E_2I_{y_2} + \frac{1}{3}E_dI_{y_d}\right)}{L^3} - \dfrac{2\sqrt{3}(E_1I_{z_1} + E_2I_{z_2} + E_dI_{z_d})}{L^3}$
$F_{1111} = F_{2222}$	$\dfrac{3\sqrt{2}}{8}(E_1A_1 - E_2A_2) + \dfrac{\sqrt{2}}{L^2}(E_1I_{y_1} - E_2I_{y_2}) + \dfrac{\sqrt{2}}{2L^2}(E_1I_{z_1} - E_2I_{z_2})$

F_{1122}	$\frac{\sqrt{2}}{8}(E_1A_1 - E_2A_2) - \frac{\sqrt{2}}{L^2}(E_1I_{y_1} - E_2I_{y_2}) - \frac{\sqrt{2}}{2L^2}(E_1I_{z_1} - E_2I_{z_2})$
F_{1212}	$\frac{\sqrt{2}}{8}(E_1A_1 - E_2A_2) + \frac{\sqrt{2}}{L^2}(E_1I_{y_1} - E_2I_{y_2}) + \frac{\sqrt{2}}{2L^2}(E_1I_{z_1} - E_2I_{z_2})$
$F_{3311} = F_{3322} = F_{3312}$	0
$D_{1111} = D_{2222}$	$\frac{3L}{8\sqrt{3}}(E_1A_1 + E_2A_2) + \frac{3\left(G_1J_1 + G_2J_2 + \frac{1}{9}G_dJ_d\right)}{4\sqrt{3}L} + \frac{7\left(E_1I_{y_1} + E_2I_{y_2} + \frac{3}{7}E_dI_{y_d}\right)}{4\sqrt{3}L} + \frac{2\left(E_1I_{z_1} + E_2I_{z_2} + \frac{1}{12}E_dI_{z_d}\right)}{\sqrt{3}L}$
D_{1122}	$\frac{L}{8\sqrt{3}}(E_1A_1 + E_2A_2) - \frac{3\left(G_1J_1 + G_2J_2 + \frac{1}{9}G_dJ_d\right)}{4\sqrt{3}L} - \frac{3\left(E_1I_{y_1} + E_2I_{y_2} - \frac{1}{3}E_dI_{y_d}\right)}{4\sqrt{3}L} - \frac{E_dI_{z_d}}{6\sqrt{3}L}$
D_{1212}	$\frac{L}{8\sqrt{3}}(E_1A_1 + E_2A_2) + \frac{3\left(G_1J_1 + G_2J_2 + \frac{1}{9}G_dJ_d\right)}{4\sqrt{3}L} + \frac{5\left(E_1I_{y_1} + E_2I_{y_2} + \frac{1}{5}E_dI_{y_d}\right)}{4\sqrt{3}L} + \frac{\left(E_1I_{z_1} + E_2I_{z_2} + \frac{1}{6}E_dI_{z_d}\right)}{\sqrt{3}L}$

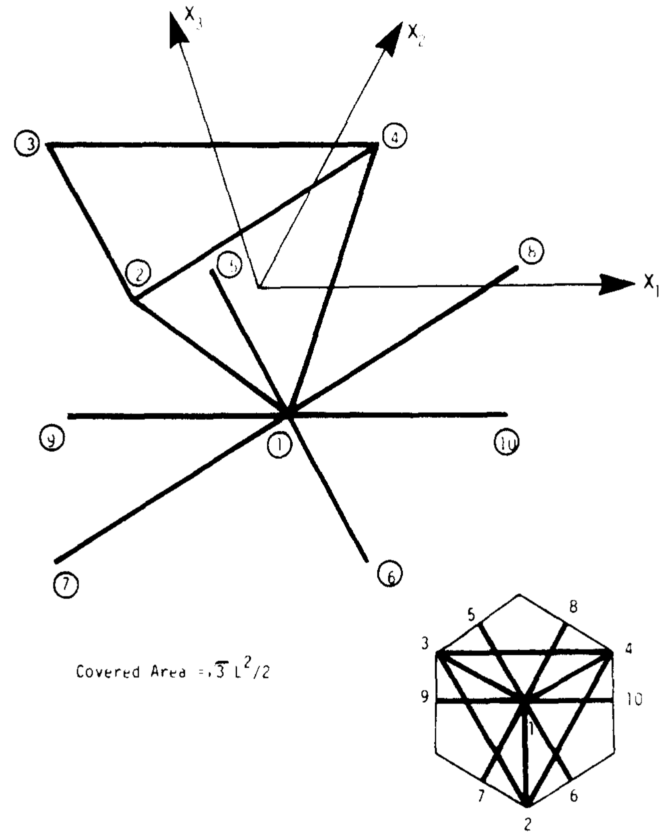

Fig. 6. The repeating element for the double layer tetrahedral grids.

continua. The transition from the discrete system to the continuous media was done by expanding the displacements and the rotations of the nodal points of the representative repeating element in a Taylor series about a suitable chosen origin. Basic kinematic assumptions were introduced to insure that the assumptions used in deriving the governing equations of the modeled plate continuum were satisfied.

The stiffness coefficients and the bending rigidities of the derived plate continuum were expressed in terms of the geometrical and material properties of the lattice elements. The geometrical properties included the cross-sectional area, the two principal cross-section bending rigidities and the cross section torsional rigidity of the elements. The material properties consisted from the modulus of elasticity and the shear modulus of the materials forming the elements.

To illustrate our method, we derived the equivalent shear deformation plate continuum for the double layered tetrahedral grids. We notice that the characteristics describing this continuum (Table 3) constitute a modification of our previously reported results[2]. This is reflected by the appearance of the bending and torsional rigidities of the lattice elements, in the effective properties of the equivalent plate continuum. Examination of the results in Table 3 indicates that $C_{1212} = (C_{1111} - C_{1122})/2$, $F_{1212} = (F_{1111} - F_{1122})/2$ and $D_{1212} = (D_{1111} - D_{1122})/2$ and hence the octetruss is transversely isotropic, as is expected.

Remark. The algebraic expressions in this analysis were obtained using the algebraic programming system Reduce 2 written by Hearn[19].

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APPENDIX

The strain energy for a repeating element governed by the shear deformation plate theory is

$$\begin{aligned} U = \frac{1}{2}a[&A_{\alpha\beta\gamma\rho}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho} + 2A_{\alpha\beta33}\varepsilon_{\alpha\beta}\varepsilon_{33} + 2F_{\alpha\beta\gamma\rho}\varepsilon_{\alpha\beta}K_{\gamma\rho} \\ &+ 2A_{\alpha\beta\gamma3}\varepsilon_{\alpha\beta}(2\varepsilon_{\gamma3}) + 2F_{\alpha\beta33}\varepsilon_{33}K_{\alpha\beta} + D_{\alpha\beta\gamma\rho}K_{\alpha\beta}K_{\gamma\rho} \\ &+ A_{\alpha3\beta3}(2\varepsilon_{\alpha3})(2\varepsilon_{\beta3}) + A_{3333}\varepsilon_{33}^2] \end{aligned} \tag{A1}$$

where a is the platform area of the repeating element. If the transverse normal stress resultant is neglected, then the transverse normal strain ε_{33} is given by

$$\varepsilon_{33} = -\frac{A_{33\gamma\rho}}{A_{3333}}\varepsilon_{\gamma\rho} - \frac{F_{33\gamma\rho}}{A_{3333}}K_{\gamma\rho}. \tag{A2}$$

The strain energy for the repeating element of the reduced model becomes

$U = \frac{1}{2}a[\overline{A_{\alpha\beta\gamma\rho}}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho}$	extensional strain energy
$+ \overline{D_{\alpha\beta\gamma\rho}}K_{\alpha\beta}K_{\gamma\rho}$	bending strain energy
$+ \overline{A_{\alpha3\beta3}}(2\varepsilon_{\alpha3})(2\varepsilon_{\beta3})$	transverse shear strain energy
$+ 2\overline{A_{\alpha3\beta\gamma}}(2\varepsilon_{\alpha3})\varepsilon_{\beta\gamma}$	transverse shear-extensional coupling
$+ 2\overline{F_{\alpha\beta\gamma\rho}}\varepsilon_{\alpha\beta}K_{\gamma\rho}]$	bending extensional coupling (A3)

(the underlined terms represent the shear deformation contribution in the governing equations) where

$$\overline{A_{\alpha\beta\gamma\rho}} = \left[A_{\alpha\beta\gamma\rho} - \frac{(A_{33\alpha\beta})(A_{33\gamma\rho})}{(A_{3333})^2}\right] \tag{A4}$$

$$\overline{F_{\alpha\beta\gamma\rho}} = \left[F_{\alpha\beta\gamma\rho} - \frac{(A_{33\alpha\beta})(F_{33\gamma\rho})}{(A_{3333})^2}\right] \tag{A5}$$

$$\overline{D_{\alpha\beta\gamma\rho}} = \left[D_{\alpha\beta\gamma\rho} - \frac{(F_{33\alpha\beta})(F_{33\gamma\rho})}{(A_{3333})^2}\right]. \tag{A6}$$